THEORETICAL DETERMINATION OF THE SPREAD IN THE STRENGTH OF FIBERGLASS

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It is well known that structural defects lead to a concentration of stresses, thereby lowering the strength of fiberglass. Since the position of these defects in actual specimens is to a large extent random, the strength of these samples fluctuates, i.e., there is a distribution associated with strength tests. This distribution, in contrast to the stress field inside the fiberglass, can easily be determined experimentally.

In this article, we present examples of the calculation of stress concentration caused by different systems of fiber ruptures and, based on the results obtained, we calculate the spread in the strength of fiberglass, in which the starting system of defects has some regularity, resulting from the appearance of a new defect at a random point. It is assumed that the fiberglass consists of alternating fibers and bonding layers, and in addition, that the fibers function only by stretching and compressing, while the bonding agent functions only in shear.

1. **Concentration of Normal Stresses for Various Defect Distributions.** Let us assume that the specimen is stretched to infinity by a constant stress σ and that the stress at a point at which the fiber is ruptured equals zero. For convenience, we will subtract out of the solution the stress corresponding to uniform tension. Then, for the remaining part, there are no stresses at infinity and a stress $-\sigma$ at points at which a fiber is ruptured.

If there is a single rupture of a fiber, labelled by k, at an ordinate $y = y_k$ (y is measured along a fiber), then the distribution of stress in the j-th fiber is given by the formula [1]

$$\sigma_j(\eta) = -\frac{\sigma}{2}\int_0^{\pi}\exp\left(-2\beta\,|\eta-\eta_k|\sin\frac{s}{2}\right)\sin\frac{s}{2}\cos(j-k)\,s\,ds. \tag{1.1}$$

Here $\eta = y/\sqrt{Hh}$ is the dimensionless coordinate along a fiber; h and H are the dimensions of a fiber and of the bonding agent layer in a direction perpendicular to the fiber; $\beta^2 = G/E$, where E and G are Young's and shear modulus of the fiber and the bonding agent, respectively. In particular,

$$\sigma_j(\eta_k) = \frac{\sigma}{4(k-j)^2-1},\quad \sigma_{k\pm1}(\eta_k) = \sigma/3,$$

i.e., due to proximity to a single rupture, a fiber is subjected to a stress that is 1/3 greater than normal (taking into account the tension at infinity).

If there are several fiber ruptures at points η_m^n (m is the number of ruptured fiber, n is the number of the rupture on the m-th fiber), then in order to calculate the stress field we can use the superposition principle. It is evident that $\sum\limits_m\sum\limits_n c_m^n\sigma_j(\eta)$, where c_m^n is a constant and $\sigma_j(\eta)$ is obtained from (1.1), satisfies the equation of equilibrium for the fiberglass. The values of c_m^n are chosen from the conditions of equality of the stress $-\sigma$ at rupture points. From here, we obtain a system of linear algebraic equations for finding c_m^n

$$\sum_m\sum_n c_m^n\frac{1}{2}\int_0^{\pi}\exp\left(-2\beta\,|\eta_m^n-\eta_\mu^\nu|\sin\frac{s}{2}\right)\sin\frac{s}{2}\cos(m-\mu)\,s\,ds = 1. \tag{1.2}$$

The pairs of indices m, n and μ, ν pass through the rupture points.

The equations for equilibrium of fiberglass (see [1]) are differential-difference analogs of the Laplace equation. For this reason, it may be shown that the maximum principle is valid for them, i.e., the stresses cannot attain minimum and maximum values outside the neighborhood of the fiber ruptures (the coordinates of points with extremal stresses are the same as the rupture points, while the fiber numbers differ by unity).

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Let us examine several particular cases such that due to symmetry in the positioning of the ruptures, all rupture points are equivalent, and for this reason, all c_m^n in (1.2) are identical and equal c. Then, in each case in (1.2), only a single equation remains.

A. Periodic System of Ruptures, Positioned along a Straight Line Perpendicular to the Fibers. In this case $\eta_m^n = \eta_\mu^\nu = 0, m, \mu = \ldots, -2p, -p, 0, p, 2p,$ and p is some positive integer, $p > 1$. The system (1.2) takes the form

$$c \sum_{m=-\infty}^{\infty} \frac{1}{2} \int_0^{\pi} \sin \frac{s}{2} \cos p (m - \mu) s ds = 1$$

(μ is an integer), from where,

$$\frac{1}{c} = \sum_{r=-\infty}^{\infty} \int_0^{\pi/2} \sin s \cos 2prs ds = \frac{\pi}{2p} \operatorname{ctg} \frac{\pi}{2p}.$$

In deriving the last relation, we used the formula [2]

$$\sum_{k=-\infty}^{\infty} \cos kps = \frac{2\pi}{p} \sum_{k=-\infty}^{\infty} \delta\left(s - \frac{2\pi k}{p}\right) \tag{1.3}$$

(δ is the Dirac function). Thus, the stress in the fibers along the rupture line equals

$$\frac{\sigma_j(0)}{\sigma} = \frac{2p}{\pi} \operatorname{tg} \frac{\pi}{2p} \sum_{m=-\infty}^{\infty} \int_0^{\pi/2} \sin s \cos 2(pm - j) s ds = \sin^2 \frac{\pi}{2p} \Big/ \left[\sin \frac{\pi(2j+1)}{2p} \sin \frac{\pi(2j-1)}{2p}\right].$$

In a fiber neighboring a ruptured fiber (j = 1), we have

$$\frac{\sigma_1(0)}{\sigma} = \sin \frac{\pi}{2p} \Big/ \sin \frac{3\pi}{2p}. \tag{1.4}$$

For p = 2 (every other fiber is ruptured),

$$\frac{\sigma_j(0)}{\sigma} = \left(2 \sin \frac{\pi(2j+1)}{4} \sin \frac{\pi(2j-1)}{4}\right)^{-1} = \begin{cases} 1 & \text{on unruptured fibers} \\ -1 & \text{on ruptured fibers}. \end{cases}$$

Therefore, the unruptured fibers are subjected to a stress that is twice the normal value. This result can also be obtained without using (1.4) from the conditions of equilibrium and the symmetry of the problem.

Table 1 shows the values of $\sigma_1(0)/\sigma$, computed from formula (1.4) for several values of p. Thus, for $p \geqslant 6$, the stress concentration created by the periodic system of defects near a defect is practically the same as in the presence of a single rupture.

B. Periodic System of Ruptures Situated along a Fiber. In this case, in (1.2), $m = \mu = 0$, $\eta_0^n = nL$, $\eta_0^\nu = \nu L$, n and ν are integers and

$$\frac{1}{c} = \sum_{n=-\infty}^{\infty} \int_0^{\pi/2} \exp(-2\beta L |n - \nu| \sin s) \sin s ds = \int_0^{\pi/2} \left[1 + 2 \sum_{n=1}^{\infty} \exp(-2\beta \times \right.$$
$$\left. \times L n \sin s)\right] \sin s ds = \int_0^{\pi/2} \operatorname{cth}(\beta L \sin s) \sin s ds.$$

The stress in the fibers along the rupture line equals

$$\frac{\sigma_j(0)}{\sigma} = -\frac{\int_0^{\pi/2} \operatorname{cth}(\beta L \sin s) \sin s \cos 2js ds}{\int_0^{\pi/2} \operatorname{cth}(\beta L \sin s) \sin s ds}. \tag{1.5}$$

In the limiting case $\beta L \ll 1$, we use formula (1.411.8) from [3]:

$$\operatorname{cth} x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + O(x^5) \quad \text{for} \quad x \to 0,$$

TABLE 1

p	$\sigma_1(0)/\sigma$
2	1
3	0,50
4	0,41
6	0,37
∞	0,33

and after computing all required integrals we obtain

$$\frac{\sigma_j(0)}{\sigma}=\frac{(\beta L)^2\delta_{1j}-0.066(\beta L)^4(\delta_{1j}-0.25\delta_{2j})}{12+2(\beta L)^2-0.1(\beta L)^4}+O(\beta^5L^5)$$

for $\beta L\to 0$ (δ_{ij} is the Kronecker symbol). In particular,

$$\frac{\sigma_1(0)}{\sigma}\approx(\beta L)^2\frac{1-0.066(\beta L)^2}{12+2(\beta L)^2-0.1(\beta L)^4}.$$

The more often the defects appear in the structure, the lower the stress concentration and, for $\beta L\to 0$, it completely vanishes.

C. **Rectangular Grid of Defects.** Let the spacing of the grid equal L along a fiber and a positive integer p ($p\geqslant 2$) in a direction perpendicular to the fibers. In this case, in (1.2) the following values must be used:

$$\eta_m^n=nL,\quad \eta_\mu^\nu=\nu L,\quad m,\mu=\ldots,-2p,-p,0,p,2p,\ldots,$$
$$n,\nu=\ldots,-3,-2,-1,0,1,2,3,\ldots,\quad c_m^n=c.$$

Then, using (1.3), we obtain

$$\frac{1}{c}=\int_0^{\pi/2}\sum_{k=-\infty}^{\infty}\cos 2kps\sum_{l=-\infty}^{\infty}\exp(-2\beta Ll\sin s)\sin s\,ds=$$

$$=2\pi\int_0^{\pi/2}\sum_{k=-\infty}^{\infty}\delta(2ps-2\pi k)\operatorname{cth}(\beta L\sin s)\sin s\,ds=\frac{\pi}{p}\sum_{k=1}^{[p/2]}\operatorname{cth}\left(\beta L\sin\frac{\pi k}{p}\right)\sin\frac{\pi k}{p}\left(1-\frac{1}{2}\delta_{k,p/2}\right)$$

([p/2] is the integer part of a number);

$$\frac{\sigma_j(0)}{\sigma}=-c\frac{\pi}{p}\sum_{k=1}^{[p/2]}\operatorname{cth}\left(\beta L\sin\frac{\pi k}{p}\right)\sin\frac{\pi k}{p}\cos\frac{2\pi kj}{p}\left(1-\frac{1}{2}\delta_{k,p/2}\right).\qquad(1.6)$$

Let the defects be positioned as densely as possible in a direction perpendicular to the fibers ($p = 2$). Then the stress in the unruptured fibers, given by (1.6), equals σ, as it should be. For $p = 3$, we have, correspondingly, $\sigma/2$. These results can be obtained from the symmetry and conditions of equilibrium without solving the equations. For $p\to\infty$, the sums in (1.6) become integrals and in the limit we obtain formula (1.5). For $\beta L\to\infty$, we obtain the formulas for a single series of defects from example (A). If in (1.6) βL is taken to the limit zero, then, replacing cth α by $1/\alpha$ and summing the cosines, we obtain that the stresses are uniformly distributed between the fibers

$$\sigma_j(0)/\sigma=1/(p-1).$$

D. **Two Defects.** In (1.2), we substitute $m=0,\ \eta_0=0,\ \mu=p,\ \eta_p=L,\ c_0=c_1=c$,

$$\frac{1}{c}=\int_0^{\pi/2}[1+\exp(-2\beta L\sin s)\cos 2ps]\sin s\,ds.$$

The stress at an arbitrary point of the j-th fiber is given by the formula

$$\frac{\sigma_j(\eta)}{\sigma}=-c\int_0^{\pi/2}[\exp(-2\beta|\eta|\sin s)\cos 2js+\exp(-2\beta|L-\eta|\sin s)\cos 2(p-j)s]\sin s\,ds.\qquad(1.7)$$

If two defects are located next to each other (p = 1, L = 0), then $\sigma_{-1}(0)/\sigma = \sigma_2(0)/\sigma = 3/5$. The riskiest positioning of the defects (p = 2, L = 0) leads to a concentration $\sigma_1(0)/\sigma = 5/7$.

2. Spread in the Magnitude of the Strength of Fiberglass. Using the formulas from Part 1, let us examine the probability that any particular value of strength will be observed in tension tests on fiberglass. Let us assume that the specimen initially has some number of defects (fiber ruptures). Let a fiber have a rupture strength σ_*. Further, let σ_0 be the smallest tensile stress at infinity for which the stress at some point in one of the fibers attains the threshold strength σ_*. Then, we will identify the strength of a specimen with a given system of defects by the ratio σ_0/σ_*. It is evident that the strength determined in this manner does not depend on σ_*, but is determined only by the geometry of the distribution of defects.

Let us now consider the following series of tests. Let us assume that an arbitrarily positioned defect is added to the initial system of defects and let us calculate the strength of the specimen in this case. We force the additional defect to pass over the entire specimen. Each position of the defect will correspond to its own strength value. Thus, we will obtain a distribution of the number of tests with respect to the magnitude of the strength of the specimen.

Let us assume that the starting system of defects forms a rectangular grid (example C). According to the maximum principle, the strength of the specimen in this case is determined by formula (1.6) with j = 1. Now, let a new defect appear at an arbitrary point in the elementary cell of the initial grid. In order to calculate precisely the strength of the specimen in this case, it is necessary to solve the system (1.2), in which now, for such a configuration of defects there is no symmetry and the number of different c_m^n in general case is infinite. In this form, this system cannot be solved, and for this reason, in what follows, in calculating the strength of a specimen, only the interaction of a new defect with one or four of the nearest defects in the starting grid will be taken into account (1.7), and in the second, from a similar formula for five defects, which can be obtained from (1.2), but is not written out here due to its cumbersomeness. It is clear that due to the symmetry it is enough to force the new defect to pass only one fourth of an elementary cell in the starting grid. From the results of part one, it follows that the error introduced into the distribution by substituting the entire grid by a single or several nearest defects decreases with increasing grid spacing.

Figure 1 illustrates the results of the tests assuming that a newly formed defect interacts only with the four nearest ruptures in the starting grid. The abscissa axis shows the strength of the specimen and the ordinate axis shows the fraction of the tests in which a given strength level was observed. The area under the curves equals unity. The vertical columns include graphs obtained with the same grid spacing in a direction perpendicular to the fibers p, but for different values of the grid spacing along the fibers L, while the horizontal rows show graphs for constant values of L. We note immediately that in taking into account only the interaction of the new defect with a single nearest defect in the starting grid the results coincide, aside from small quantitative differences, with those presented in Fig. 1. As follows from (1.1) and (1.2), the stress around a rupture rapidly decreases with increasing distance from the rupture, and for this reason, two ruptures positioned at a large distance have a weak effect on each other and the strength of a specimen with such defects is close to 3/4 (strength of the specimen with a single defect). Since an increasing percentage of new defects is situated fairly far away from all the initial defects as the spacing of the starting grid increases, an increasing percentage of tests must give strength values around 3/4. Therefore, a peak that increases with increasing spacing p and L is observed in the graphs. The second peak observed for small values of L with high values of strength corresponds to the appearance of an initial rupture in an already ruptured fiber.

It would be interesting to determine how further rupture occurs in the sample, i.e., will maximum stresses be obtained in the vicinity of the new defect or in the vicinity of one of the initial defects? Such a test was carried out for the case when the new defect interacts with four of the initial defects. It turned out that if the new defect is situated on one of the initial unruptured fibers, but not too far away from one of the defects in the grid, then further rupture occurs in the vicinity of the new defect; if, on the other hand, the new defect is situated far away from all the initial defects or on an already ruptured fiber, then further rupturing occurs next to one of the initial defects.

In order to refine the model used, we can attempt to take into account in some way the random (and not regular, as in the computations) nature of the distribution of the initial defects. This can be done, for example, by averaging the results of tests according to several of the initial systems of defects with different parameters p and L. However, it is clear that such averaging will not give a qualitatively new strength distribution. It is natural to assume that the probability for the appearance of an additional defect at a given point depends on the stress created at this point by the initial system of defects. A calculation was carried out assuming that the probability for the appearance of a defect is proportional to the k-th power of the initial stress ($0 \leqslant k \leqslant 5$, p = 8, L = 8, and it was assumed that the new defect interacts with four nearest defects). The qualitative nature of the distribution in this case also remained as previously with small quantitative changes.

Up to now, dynamic effects were not taken into account, i.e., it was assumed that after the appearance of an additional defect in the specimen an equilibrium static stress distribution corresponding to the new configuration of defects is established immediately. However, in reality, when a fiber suddenly ruptures, the dynamic stress concentration near a defect considerably exceeds the static concentration [4]. For this reason, it is natural to attempt to take into account dynamic overloading in studying the strength of a specimen.

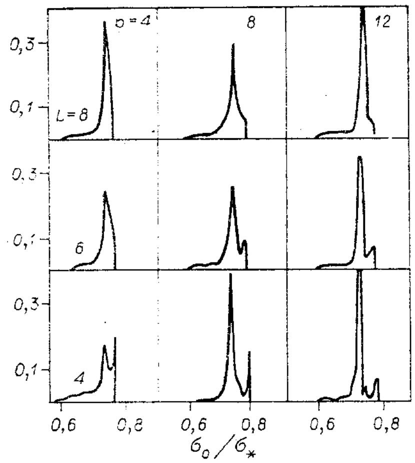

Fig. 1

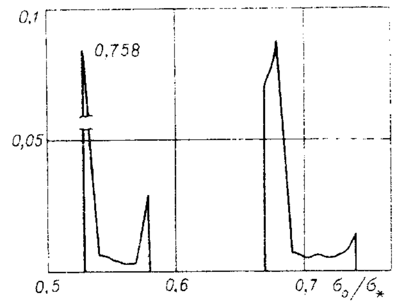

Fig. 2

In order to solve exactly the dynamic problem of the sudden appearance of a new defect, it is necessary to take into account diffraction of elastic waves, caused by the appearance of the new defect, by defects in the starting system. However, this problem is very complicated, and it has not yet been solved. For this reason, in the calculations, the true dynamic stress field was replaced by a superposition of the dynamic field of an isolated defect (see [4]) with the static field of the initial system of defects. This superposition is an exact solution to the problem at each point prior to the arrival at that point of elastic waves reflected from the initial defects. When the problem is stated in this manner, it is possible to include the entire infinite grid in the starting system of defects.

Following [5], we will adopt the following criterion for the rupturing of a fiber:

$$\frac{1}{t_*}\int\limits_{t-t_*}^{t}\sigma(\tau)\,d\tau=\sigma_*,$$

where σ_* is the rupture strength of a fiber; t_* is some characteristic averaging time. It is not useful to state the conditions for strength according to the maximum stresses, since the stresses change very rapidly with time. Small changes in the model, for example taking into account viscosity, can considerably change the maximum values of the stresses and thereby cast doubt on the conclusions obtained. Finally, it is not reasonable to make the behavior of the mechanical system depend on conditions occurring during the zeroth time interval. For this reason, in stating the conditions for strength, we average with respect to time.

Figure 2 shows the results of calculations for $p = 8$ and $L = 8$. The strength criteria were averaged according to two times for passage of shear waves in the binding agent between two neighboring fibers. Since diffraction of elastic waves on initial defects is not taken into account, these results are very approximate, but, apparently, they nevertheless give a qualitative description of the phenomenon. Taking the dynamic overloads into account leads, as expected, to a considerable decrease in the strength of a specimen. The curve on the right side of Fig. 2, which corresponds to higher strength values, corresponds to the appearance of an additional rupture in a fiber that has already been ruptured.

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